The 'Cut and Try' Method in the Design of the Bow [1]

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1 Introduction

In history man used bows which differ much in shape as well as applied materials. Simple bows made out of one piece of wood, straight and tapering towards the ends have been used by primitives in Africa, South America and Melanesia. In the famous English longbow the different properties of the sapwood and heartwood were deliberately put to use. Eskimos used wood together with cords plaited of animal sinews and lashed to the wooden core at various points. The Angular bow found in Egypt and Assyria are examples of composite bows. In these bows more than one material was used. In Asia the bow consisted of wood, sinew and horn. These bows reached their highest development in India, in Persia and in Turkey. In the 1960's composite bows of maple and glass fibres, or later carbon fibres, imbedded in strong synthetic resin were designed. Today almost all bows seen at target archery events are of this type of bow.

Bowyers (manufacturers of archery equipments) relied for the design of the bow heavy upon experience. The performance of the bow was improved by 'try and cut' method. In the 1930's bows and arrows became the object of study by scientists and engineers, Hickman, Klopsteg and Nagler, see [4] and [5]. Their work influenced strongly the design and construction of the bow and arrow. Experiments were performed to determine the influence of different parameters. They also made mathematical models. As part of modelling simplifying assumptions were made. Hence only bows with specific features could be described. In [6] and [7] we dealt with the mechanics of the different types of bow: non-recurve, static-recurve and working-recurve bows. The developed mathematical models are much more advanced, so that more detailed information was obtained giving a better understanding of the action of rather general types of bow. In Section 2 of this paper the problem is formulated. All design parameters are charted accurately and quality coefficients are identified. The importance of the application of dimensional analysis is emphasized. In Section 3 the performance of different types of bow are compared. Roughly speaking the design parameters can be divided into two groups. One determines the mechanical performance of the bow. Within certain limits, these parameters appear to be less important as is often claimed. The other group of parameters concerns the strength of the materials and the way these materials are used in the construction of the bow. It turns out that the application of better materials and that more of this material is used to a larger extent, contribute most to the improvement of the bow.

[1] *Engineering Optimization in Design Processes*, Eschenauer, H.A.; Mattheck, C.; Olhoff, N. (eds): Lecture Notes in Engineering 63, Berlin: Springer-Verlag, 283-292 (1991)

2 Formulation of the problem

In essence the bow proper consists of two elastic limbs, often separated by a rigid middle part called grip. Because the bow is usually held vertical or nearly vertical, we can speak of the upper limb and of the lower limb. The back of a limb is the side facing away from the archer, the belly the opposite side. The bow is braced by fastening a string between both ends of the limbs. The distance between the grip on the belly side and the string in that situation is called the brace height or fistmele. After an arrow is set on the string, called nocking, the archer pulls the bow from braced situation in full draw. This action is called drawing. Then, after aiming, the arrow is loosed or released, called loosing.

We are concerned with bows of which the limbs move in a flat plane, and which are symmetric with respect to the line of aim. The bow is placed in a Cartesian coordinate system $(\overline{x}, \overline{y})$, the line of symmetry coinciding with the $\overline{x}$-axis and the origin O coinciding with the midpoint of the bow, see Figure 1. We assume the limbs to be inextensible and that the Euler-Bernoulli beam theory holds. The total length of the bow is denoted by $2\,\overline{L}$. In our theory it will be represented by an elastic line of zero thickness, along which we have a length coordinate $\overline{s}$ measured from O, hence for the upper half we have $0 \leq \overline{s} \leq \overline{L}$. This elastic line is endowed with bending stiffness $\overline{W}(\overline{s})$ and mass per unit of length $\overline{V}(\overline{s})$. The geometry of the unstrung bow is described by the local angle $\theta_0(\overline{s})$ between the elastic line and the y-axis, the subscript 0 indicates the unstrung situation. $\overline{L}_0$ is the half length and $2\,\overline{m}_g$ the mass of the grip. The length of the unloaded string is denoted by $2\,\overline{l}_0$, its mass by $2\,\overline{m}_s$. We assume that the material of the string obeys Hooke's law, the strain stiffness is denoted by U_s. Note that whether the length of the string or the brace height denoted by $|OH|$ fixes the shape if the bow in braced situation.

The classification of the bow we use, is based on the geometrical shape and the elastic properties of the limbs. The bow of which the upper half is depicted in Figure 1.a is called a non-recurve bow. These bows have contact with the string only at their tips ($\overline{s} = \overline{L}$) with coordinates $(\overline{x}_t, \overline{y}_t)$. There may be concentrated masses $\overline{m}_t$ with moment of inertia $\overline{J}_t$ at each of the tips, representing for instance horns used to fasten the string.

In the case of the static-recurve bow, see Figure 1.b, the outermost parts of the limbs are stiff. These parts are called ears. Its mass and moment of inertia with respect to the centre of gravity of the ear $(\overline{x}_{cg}, \overline{y}_{cg})$ are denoted by $\overline{m}_e$ and $\overline{J}_e$, respectively. The flexible part $\overline{L}_0 \leq s \leq \overline{L}_2$ is called the working part of the limb. In the braced situation the string rests upon string-bridges, see Figure 1.b. These string-bridges are fitted to prevent the string from slipping past the limbs. The place of the bridge of the upper limb is referred to as $(\overline{x}_b, \overline{y}_b)$.

With a working-recurve bow the parts near the tips are elastic and bend during the final part of the draw, see Figure 1.c. When drawing such a bow the length of contact between string and limb decreases gradually until the point where the string leaves the limb, denoted by $\overline{s} = \overline{s}_w$, coincides with the tip $\overline{s} = \overline{L}$ and remains there during the final part of the draw. In Figure 1 the bow is pulled by the force $\overline{F}(\overline{b})$, where the middle of the string has the $\overline{x}$-coordinate $\overline{b}$. To each bow belongs a value $\overline{b} = |\overline{OD}|$ for which it is called fully drawn indicated by a subscript 1. The force $\overline{F}(|\overline{OD}|)$ is called the weight of

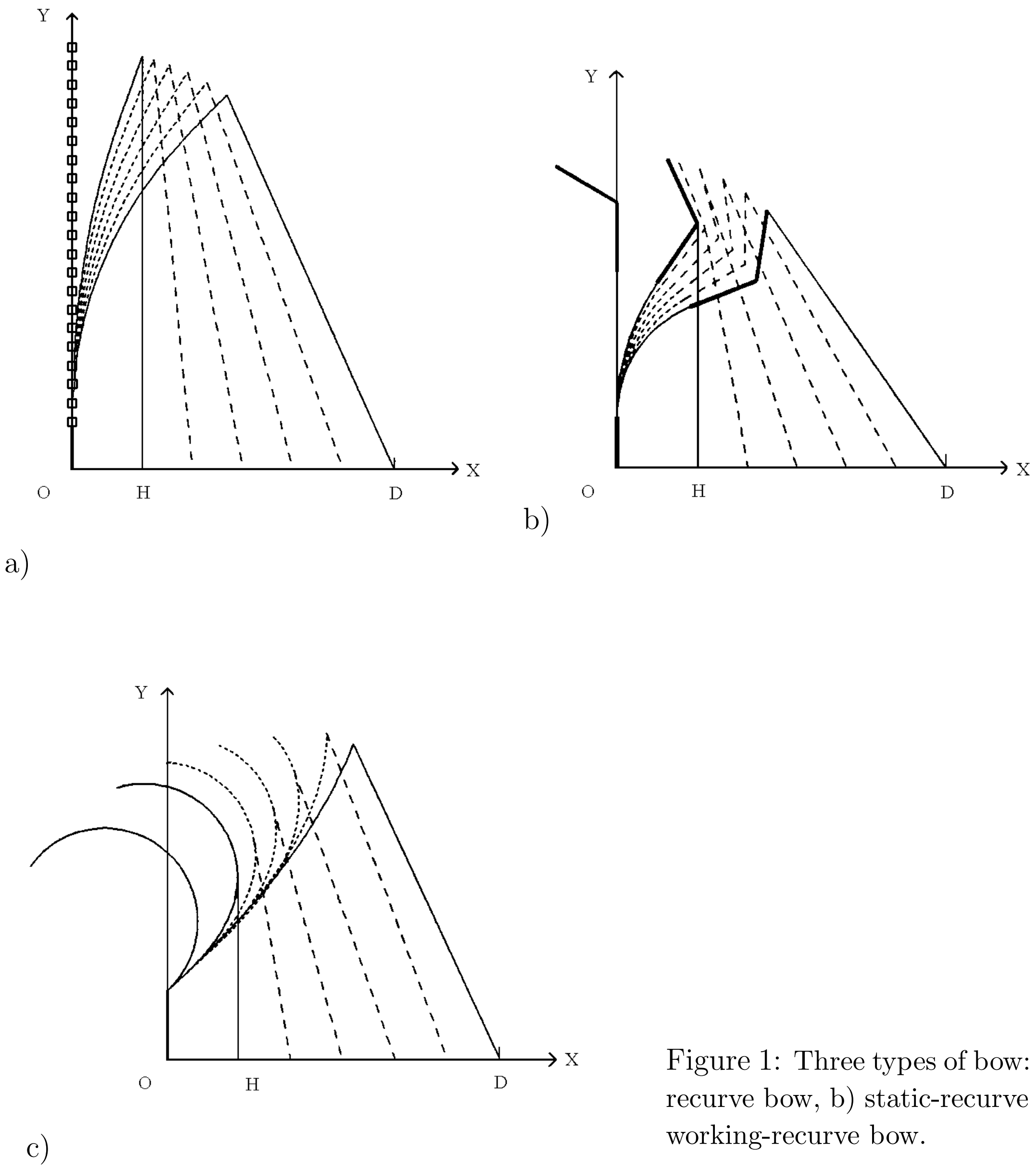

Figure 1: Three types of bow: the a) non-recurve bow, b) static-recurve bow and c) working-recurve bow.

the bow and the distance $|\overline{OD}|$ is its draw. By releasing the drawn string at time $\overline{t} = 0$ and holding the bow at its place, the arrow, represented by a point mass $2\,\overline{m}_a$ is propelled. The arrow leaves the string when the acceleration of the midpoint of the string becomes negative. This moment is denoted by $\overline{t}_l$ and the muzzle velocity of the arrow is referred to as $\overline{c}_l$.

A shorthand notation for a bow and arrow combination is introduced with

$$\overline{B}(\overline{L}, \overline{L}_0, \overline{W}(\overline{s}), \overline{V}(\overline{s}), \theta_0(\overline{s}), \overline{m}_a, \overline{m}_t, \overline{J}_t, \overline{m}_e, \overline{J}_e, \overline{m}_g, \overline{x}_{cg0}, \overline{y}_{cg0}, \overline{x}_{b_0}, \overline{y}_{b_0}, \overline{x}_{t_0}, \overline{y}_{t_0}, \overline{L}_2, \overline{U}_s, \overline{m}_s, |\overline{OH}| \text{ or } \overline{l}_0; |\overline{OD}|, \overline{F}(|\overline{OD}|), \overline{m}_b), \tag{1}$$

where $\overline{m}_b$ is the mass of one limb excluding the mass of the grip.

Note that the last two mentioned parameters are added to the list artificially. This implies that both functions $\overline{W}(\overline{s})$ and $\overline{V}(\overline{s})$ are constrained. We consider the values of these functions for $\overline{s} = \overline{L}_0$ to be already fixed by both constraints. The first constraint concerning the weight, is an implicit relationship between a number of parameters of which $\overline{W}(\overline{s})$ is one of them, and the weight $\overline{F}(|\overline{OD}|)$ of the bow. The second constraint is just

$$\overline{m}_b = \int_{\overline{L}_0}^{\overline{L}_2} \overline{V}(\overline{s}) d\overline{s} + \overline{m}_e \; . \tag{2}$$

and for a given mass of the bow the value $\overline{V}(\overline{L}_0)$ is derived easily. This shows that both functions are considered to be the product of a function $\overline{W}(\overline{s})/\overline{W}(\overline{L}_0)$ and $\overline{V}(\overline{s})/\overline{V}(\overline{L}_0)$ of the length coordinate $\overline{s}$ into IR and a parameter $\overline{W}(\overline{L}_0)$ and V$(\overline{L}_0)$ with dimensions.

2.1 Dimensional analysis

In this paragraph the 24 parameters of Equation (1) are considered as elements of a dimensional space p, spanned by a fixed system of units E_1 = length in cm, E_2 = force in kgf, E_3 = mass in kg and E_4 = time in 0.03193 s (see later on). According to the Π-theorem one can write Equation (1) in the form with the dimensionally independent parameters $|\overline{OD}|$, $\overline{F}(|\overline{OD}|)$ and $\overline{m}_b$, referred to as $\overline{A}_1$, $\overline{A}_2$ and $\overline{A}_3$ (dimensional base) and dimensionally dependent parameters $\overline{L}, \cdots, |\overline{OH}|$ or $\overline{l}_0$, referred to as $\overline{B}_1, \cdots, \overline{B}_{21}$. With $j = 1(1)21$ and $i = 1(1)3$ we have,

$$\overline{B} = B(B_1, \cdots, B_{21}) \prod_{i=1}^{3} \overline{A}_i^{a_i} \quad , \quad \overline{B}_j = B_j \prod_{i=1}^{3} \overline{A}_i^{a_{ji}} \quad , \quad \overline{A}_i = A_i \prod_{k=1}^{4} E_k^{z_{ik}} \; , \tag{3}$$

where $a_{ji}, z_{ik}, a_i \in IR$, and $B_j \in IR^+$, where R is the set of real numbers and $\mathbb{R}^+$ the set of positive numbers, see [10, page 951]. We have for example

$$\overline{L} = L\,|\overline{OD}| \quad , \quad \overline{K} = K\,\overline{F}(|\overline{OD}|) \quad , \quad \overline{m}_a = m_a\,\overline{m}_b \; , \tag{4}$$

where $\overline{K}$ is the force in the string, and for the functions of the length coordinate $\overline{s}$

$$\overline{W}(\overline{s}) = W(s)\,|\overline{OD}|^2\,\overline{F}(|\overline{OD}|) \quad , \quad \overline{V}(\overline{s}) = V(s)\,\overline{m}_b/|\overline{OD}| \; . \tag{5}$$

Observe that these functions of $\overline{s}$ are also transformed to functions of the dimensionless length coordinate s. Also the angle $\theta(\overline{s})$ between the elastic line and the y-axis will be transformed to $\theta(s)$, where we should have used a new symbol. With respect to dimensional analysis this yields no added difficulties. Finally we have

$$|\overline{OD}| = |OD| \text{ cm} \quad , \quad \overline{F}(|\overline{OD}|) = F(|OD|) \text{ kgf} \quad , \quad \overline{m}_b = m_b \text{ kg} . \tag{6}$$

So, quantities with dimension are labelled by means of a bar "¯" and quantities without the bar are the associated dimensionless quantities. The unit of time is already fixed by the choice of the other 3 units: cm, kgf and kg. For the time t_l, the moment the arrow leaves the string, we have

$$\overline{t}_l = t_l(L, \cdots, |OH| \text{ or } l_0)\sqrt{\frac{\overline{m}_b\,|\overline{OD}|}{\overline{F}(|\overline{OD}|)}} = t_l\sqrt{\frac{m_b\,|OD|}{F(|OD|)}}\sqrt{\frac{\text{kg cm}}{\text{kgf}}} . \tag{7}$$

This means that the unit of time equals 0.03193 s.

2.2 Quality coefficients

The purpose for which the bow is used has to be considered in the definition of a cost-function which could be optimized in order to obtain the 'best' bow. However, it appears to be very difficult to define such a unique cost-function. Therefore we introduce a number of quality coefficients which can be used to judge the performance of a bow and arrow combination. The static quality coefficient q is given by

$$q = A = \frac{\overline{A}}{|\overline{OD}|\,\overline{F}(|\overline{OD}|)} , \tag{8}$$

where A is the dimensionless energy stored in the elastic parts of the bow, the working parts of the limb and the string, by deforming the bow from the braced position into the fully drawn position. The dynamic quality coefficients are the efficiency η and the muzzle velocity referred to as ν. They are defined by

$$\eta = \frac{\overline{m}_a\,\overline{c}_l^2}{\overline{A}} , \tag{9}$$

Observe that by definition these quality coefficients are dimensionless. This means that the sensitivities of these coefficients with respect to the elements of the dimensional base $|OD|$, $F(|OD|)$ and m_b can be obtained directly, without solving the governing equations of motion which constitute the mathematical model again. The selection of the dimensional base is not unique. The motivation to take draw, weight, and mass of one limb is the following. The maximum draw and weight depend on the stature of the archer. His 'span' determines the maximum draw and his strength the maximum weight, so both have physical limitations. In practice the minimum mass of one limb has technical limitations which will be the subject of the next section. In our approach the elements of the dimensionless basis are selected based on the limitations which make the optimization problem well posed. Hence, the choice of the dimensional base is coupled to the formulation of the quality coefficients. The advantage of this technique is that with the comparison of different bows, taking the quantities $|OD|$, $F(|OD|)$ and m_b equal to 1, yields interpretable results for the quality coefficients.

2.3 The construction of the bow

The limbs of the bow are considered as a beam of variable cross-section $D(s)$ made out of one material with density r and Young's modulus E. With the Euler-Bernoulli hypothesis the normal stress $\sigma(s,t,r)$ depends linearly on r, the distance from the neutral axis which passes through the centroid of the cross-section. We assume that the maximum bending moment for each s as a function of time t occurs in the fully draw situation. Then we have, when A_b denotes the elastic energy in the limbs of the bow in fully drawn situation,

$$\frac{\overline{A}_b}{\overline{m}_b} = \frac{2\int_{\overline{L}_0}^{\overline{L}} \int_{\overline{D}(\overline{s})} \frac{1}{2}\,\frac{\overline{\sigma}_1^2(\overline{s},\overline{r})}{\overline{E}}\, d\overline{D}\, d\overline{s}}{\int_{\overline{L}_0}^{\overline{L}} \overline{\rho}\overline{C}(\overline{s})\, d\overline{s}}\ , \tag{10}$$

where $C(s)$ is the area of the cross-section. The stress σ_1 is the resulting normal stress due to the bending moment in the fully drawn bow, indicated by the subscript 1.

We define now two useful quality coefficients

$$\overline{\delta}_{bv} = \frac{\overline{\sigma}_w^2}{2\,\overline{\rho}\overline{E}}\quad ,\quad a_D = \frac{\overline{A}_b}{2\,\overline{m}_b\overline{\delta}_{bv}}\ , \tag{11}$$

where $\overline{\sigma}_w$ is the working stress of the material, equal to the yield point or the ultimate strength divided by factors of safety. The quantity $\overline{\delta}_{bv}$ is the amount of energy per unit of mass which could be stored in the material. In Table 1 an indication of this quantity of some materials used in making bows is given.

The dimensionless coefficient a_D is generally smaller than 1 for two reasons. First, the tensile and compression stresses in the outermost fibres of the limb may be less than the working stress in the fully drawn bow. With the design of the limbs one has to assure stability of the limb, without tendency to twist or distort laterally when the bow is drawn. In practice this is accomplished by the requirement that the width of a limb may not become smaller than the thickness. Not all material near to the tips is used then to the full extent. The stress in the fibres near the neutral axis is smaller than the working stress and this reduces the coefficient a_D too. Suppose the bow is in a homogeneous stress-state, then

$$a_D = \frac{\int_{\overline{L}_0}^{\overline{L}} \frac{\overline{I}}{\overline{e}^2(\overline{s})}\, d\overline{s}}{\int_{\overline{L}_0}^{\overline{L}} \overline{C}(\overline{s})\, d\overline{s}}\ , \tag{12}$$

Table 1: Mechanical properties and the energy per unit of mass δ_{bv} for some materials used in making bows, see also [2].

material	$\overline{\sigma}_w$ kgf/cm^2 10^2	$\overline{E}$ kgf/cm^2 10^5	$\overline{\rho}$ kg/cm^3 10^{-6}	$\overline{\delta}_{bv}$ kgf cm/kg
steel	70.0	21.0	7800	1300
sinew	7.0	.09	1100	25000
horn	9.0	.22	1200	15000
yew	12.0	1.0	600	11000
maple	10.8	1.2	700	7000
glassfibre	78.5	3.9	1830	43000

where e is the distance between the outermost fibres and the neutral axis. If the material has the same strength in tension and compression, it will be logical to choose shapes of cross-section in which the centroid is at the middle of the thickness of the limb, equal to $2\,e(s)$. $I(s)$ is the moment of inertia of the cross-section with respect to the neutral axis. The quantity $2\,I/e$ is called the section modulus. In handbooks the magnitude of the moment of inertia and the section modulus are tabulated for various profile sections in commercial use. We stick at the usage of the defined coefficient a_D which is dimensionless and follows in a straightforward manner from our statement of the problem.

We consider the quantity α_D defined by

$$\alpha_D(s) = \frac{I(s)}{C(s)e(s)^2} \quad , \quad L_0 \leq s \leq L, \tag{13}$$

for various shapes of cross-section of limbs. For a bow with similar cross-sections at different values of s we have $a_D = \alpha_D$. The English longbow possessed a D-shape cross-section, the belly side approximately formed by a semicircle and the back side being a rectangular. When the radius of the semicircle equals the half of the thickness of the limb a_D equals 0.255, so smaller than 0.333 for a rectangular and more than 0.25 for a elliptical shape. Steel bows, for instance the Seefab bow invented in the 1930's, were on the principle of a flattened tube. We assume that the inner diameter equals k times the outer diameter for any line through the centre of the ellipses. For $k = 0.9$ the magnitude of α_D becomes 0.4525, so larger than the other mentioned values, as to be expected, because relatively more material is placed near the outermost fibres.

There is still another technique to increase the value of a_D. In ancient Asiatic bows, horn and sinew, together with wood, were used on the belly and back side, respectively. Horn is a superb material for compressive strength and sinew laid in glue has a high tensile strength, see Table 1. The Young's modulus of both materials is rather small, but the permissible strain is very high. The space between the two materials near the outermost fibres is filled up with light wood, which has to withstand the shearing stresses. In modern bows horn and sinew are replaced by synthetic plastics reinforced with fibreglass or carbon. So, in composite bows not only better materials are used, but they are also used in a more profitable manner. For composite bows we define equivalent quantities for the Young's modulus and density for a simple bow which has the same mechanical action as the limb of the composite bow. If these magnitudes are substituted in the product $a_D \delta_{bv}$ it can be substantially larger than the magnitude of the product for simple wooden bows. The importance of this product follows from the equation for the stored energy A_b per mass of the limb m_b and, when we neglect the elastic energy stored in the string, the equation for the muzzle velocity

$$c_l = \sqrt{2 \frac{q}{A_b} \frac{\eta}{m_a} a_D \delta_{bv}} \ . \tag{14}$$

Hence, the muzzle velocity of an arrow depends on: the static quality coefficient q divided by A_b, the amount of elastic energy stored in the fully drawn limbs, the efficiency divided by the dimensionless mass of the arrow and finally on the product $a_D \delta_{bv}$. The first term shows that the amount of energy in the elastic parts of the bow in braced situation should be as small as possible. For q equals the energy stored in the fully drawn bow minus this energy in the braced bow. A large q implies, however, heavy limbs and this in turn implies a small m_a, because this is the arrow mass divided by the mass of one limb.

The efficiency depends largely on the masses of the arrow and string. This influence can be assessed from a simplified model of the string moving in straight lines between the points of attachment and the arrow. The resulting distribution of the kinetic energy along the string indicates that 1/3 of the mass of the string should be concentrated at the middle where the arrow meets the string. Hence,

$$\max(\eta) \approx \frac{m_a}{m_a + \frac{1}{3} m_s} \ , \tag{15}$$

is an approximation of the maximum attainable efficiency. The quotient η/m_a increases with decreasing ma and this holds probably also to a certain extent for a real bow. However, there are limits, for the arrow has to be strong enough to withstand the acceleration force. Further, every archer knows that it is not allowed to loose a fully drawn bow without an arrow. In that case the efficiency equals 0 and the bow or string can even break. For small arrow masses our assumption, the maximum bending moment to be equal to the value in the fully drawn situation, is obviously violated. There seems to be an optimum for the mass of the arrow.

In order to reduce the mass of the string the applied materials should be strong. Man-made fibres such as Dacron and Kevlar are used. The maximum force determines with the strength of the material the minimum mass of the string. This maximum force will certainly not occur in the fully drawn situation.

3 Results and conclusions

In this section we start with a sensitivity study for a straight-end bow. In our mathematical model derived in [7], the action of a bow and arrow combination is fixed by one point in a 24 dimensional parameter space. First of all we deal with the 21 dimensionless parameters. Three of them are functions, viz. $W, V, \theta : [L_0, L] \rightarrow \mathbb{R}$. These functions are written in a simple form:

$$W_n(s) = W_n(L_0)\big(\frac{(L-s)}{L-L_0}\big)^{\beta_n} \ , \ L_0 \leq s \leq \epsilon_n \ , \ W_n(s) = 1/3 \ W_n(L_0) \ , \ \epsilon_n \leq s \leq L \ , \quad (16)$$

and

$$V_n(s) = V_n(L_0)\big(\frac{(L-s)}{L-L_0}\big)^{\beta_n} \ , \ L_0 \leq s \leq \epsilon_n \ , \ V_n(s) = 1/3 \ V_n(L_0) \ , \ \epsilon_n \leq s \leq L \ , \quad (17)$$

where: $\epsilon_1 = L$, $\epsilon_n = L - (L - L_0)\ (1/3)^{1/\beta_n}$ for $n = 2, 3$; $\beta_1 = 0$, $\beta_2 = 1/2$, $\beta_3 = 1$.

The shape of the unstrung bow is given by

$$\theta_0(s) = \theta_0(L_0) + \kappa_0 \frac{s - L_0}{L - L_0} \ , \ L_0 \leq s \leq L \ . \quad (18)$$

Under this description, these functions are fixed by only three parameters b, $\theta_0(L_0)$ and κ_0. The two string- parameters, the mass m_s and the stiffness U_s, are for a particular material fixed by the number of strands. An increase of this number, n_s makes the string stiffer but also heavier. We start with a straight-end bow described by Klopsteg in [4]. This bow is referred to as the KL bow (Figure 1.a). In shorthand notation introduced in Equation (1), it is represented by

$$\mathrm{KL}(1.286, 0.1429, W, V, \theta_0 \equiv 0, 0.0769, 0, 0, 0, 0, 0, 0,$$
$$1.286, 0, 1.286, 0, 1.286, 1.286, 131, 0.0209, 0.214; 1, 1, 1) \ , \quad (19)$$

The sensitivity coefficients, i.e. the partial derivatives of the quality coefficients with respect to the design parameters, are presented in Table 2. These sensitivity coefficients were calculated with the classical approach with finite-difference approximations. We conclude that the mass of the arrow is the most important parameter for the efficiency and for the muzzle velocity. Further tip-masses should be avoided because they reduce the efficiency.

Representations of different types of bows used in the past and in our time form clusters in the parameter space. In the preceding sensitivity analysis one cluster, that for a straight-end flatbow, was analysed. In what follows we consider other types of bow: another non-recurve bow, the Angular bow, to be called the AN bow (Figure 2.a), two Asian types of static-recurve bow, to be called the PE (Figure 1.b) and TU bow (Figure 2.b) and two working-recurve bows, one with an extreme recurve, to be called the ER bow (Figure 1.c) and a modern working-recurve bow to be called the WR bow (Figure 2.c).

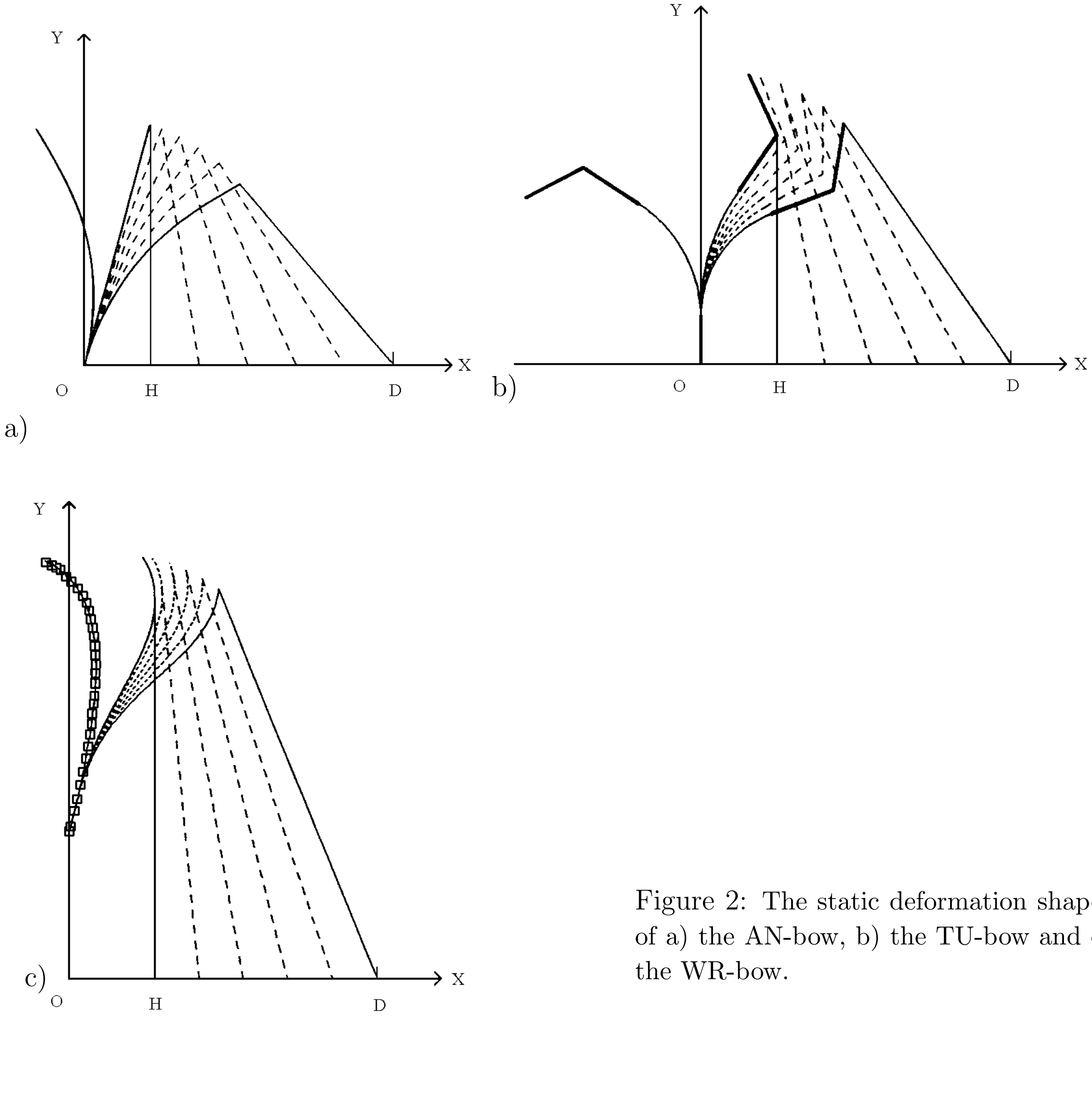

Figure 2: The static deformation shapes of a) the AN-bow, b) the TU-bow and c) the WR-bow.

Table 2: Sensitivity coefficients for the straight-end KL bow.

	L	L_0	β	$\theta_0(L_0)$	κ_0	m_a	m_t	J_t	n_s	$\|OH\|$
$W(L_0)$	4.24	-3.4	.45	2.6	1.81	.0	.0	.0	.0	-.33
$V(L_0)$	-1.4	1.4	.70	.0	.0	.0	.0	.0	.0	.0
q	.07	-.05	.0	-.15	-.18	.0	.0	.0	.0	-.39
η	-.11	.15	-.06	.46	.3	3.4	-1.1	.0	.0	.0
ν	.0	.07	.07	.25	.0	-6.7	-1.6	.0	.0	-.83

All the quality coefficients for these types of bow are shown in Table 3. The results indicate that the muzzle velocity is about the same for all types. So, within certain limits, these dimensionless parameters appear to be less important than is often claimed. The efficiency of strongly recurved bows is rather bad.

In Table 3 the values of the quality coefficients for a modern working-recurve bow are also given. The shape of the bow for a number of draw-lengths is shown in Figure 2.c. Figures 3.a and 3.b give the shape of the limb and the string, before ($0 \leq t \leq t_l$) and after ($t_l \leq t$) arrow exit. Observe the large vibrations of the string after the arrow leaves the string, which imply that the brace height has to be large.

These results show that the modern working-recurve bow is a good compromise between the non- recurve bow and the static-recurve bow. The recurve yields a good static quality coefficient and the light tips of the limbs give a reasonable efficiency. Note that the mass of the arrow of the modern working-recurve bow is smaller than the other values mentioned This accounts for a smaller efficiency, but also larger muzzle velocity.

How can these dimensionless quantities be used in the design of a bow?. In practice the manufacturer wants to design a bow with a specified draw $|OD|$ and weight $F(|OD|)$ using available materials with given Young's modulus E and density ρ. The use of Equation (5) gives a value for $I(s)$. The thickness of the limb associated with the distance between the outermost fibres and the neutral axis e is then fixed by

$$\overline{e}(s) = \frac{I(s)}{M_1(s)}\overline{\sigma}_w = \frac{W(s)}{M_1(s)}\frac{\sigma_w}{\overline{E}}|\overline{OD}| \ , \tag{20}$$

These calculations can be done after the solution of the static equations yielding $M_1(s)$. After the selection of the shape of a cross-section of the limbs $D(s)$, the width is fixed. To ensure stability this width should not be taken smaller than the thickness of the limb. The area of the cross-section $C(s)$ can then be calculated. $C(s)$ and the density ρ jointly determine the mass of the limbs m_b. This completes the design of the limbs.

Hence, for a homogeneous stressed bow two parameters given in Equation (1), the

Table 3: Dimensionless quality coefficients for a number of bows.

Bow	q	η	ν	m_a	m_s	$W(L_0)$	$V(L_0)$	A_b	$\frac{q\,\eta}{A_b\,m_a}$
KL bow	.407	.765	2.01	.0769	.0209	1.4090	1.575	0.5155	7.85
AN bow	.395	.716	1.92	.0769	.0209	0.2385	2.300	0.5493	6.70
PE bow	.432	.668	1.94	.0769	.0209	0.2304	1.867	0.5879	6.38
TU bow	.491	.619	1.99	.0769	.0209	0.1259	1.867	1.0817	3.65
ER bow	.810	.417	2.08	.0769	.0209	0.3015	2.120	1.4150	3.10
WR bow	.434	.729	2.23	.0629	.0222	2.5800	1.950	0.6930	7.25

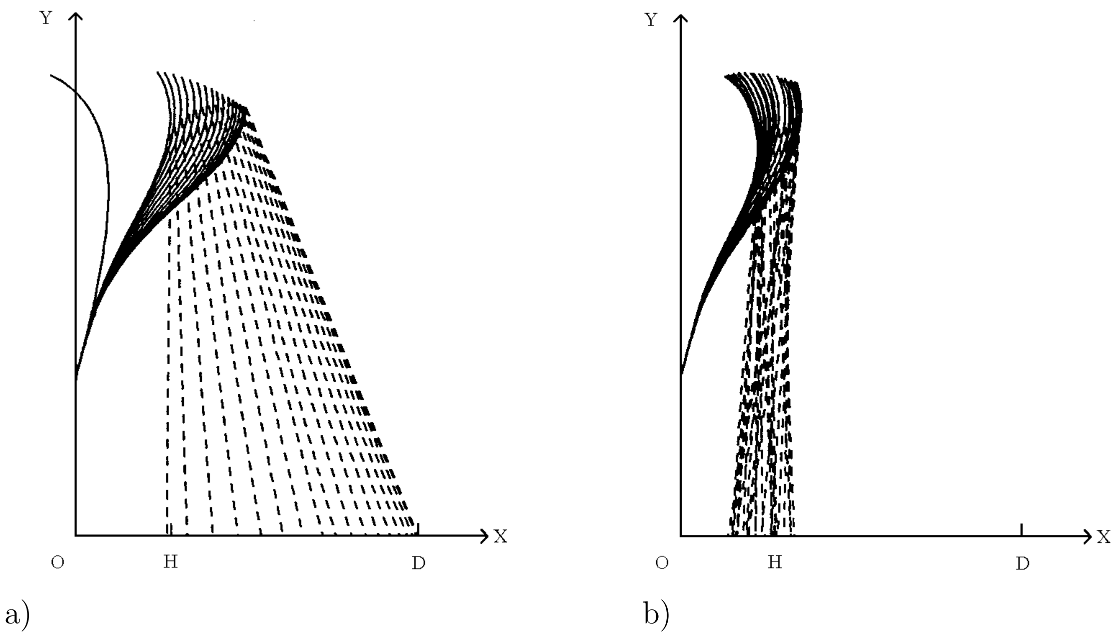

Figure 3: Dynamic deformation shapes of the WR-bow for a) $0 \leq t \leq t_l$ and b) $t_l \leq t$.

mass distribution $V(s)$ and the total mass of the limb m_b, depend on other parameters of Equation (1) and Young's modulus E, density ρ, working stress $\overline{\sigma}_w$ of the material and finally the shape of a cross-section $D(s)$. More parameters of Equation (1) are dependent in practice. The parameters concerning the ears, m_e, J_e, x_{cg}, y_{cg}, x_{b_0}, y_{b_0}, x_{t_0}, y_{t_0}, L_2, are strongly related and there is a relationship between the string parameters U_s, m_s and the strength of the material used for the string together with the maximum force in the string. This force is not known from the static calculations, so an initial guess has to be made, which must be checked after the dynamic calculations.

We considered a number of different bows used in the past and in our present. In Table 4 we give values for the parameters with dimension, weight, draw and mass of one limb, for a number of bows described in the literature. Estimations of a_D are also given. These approximations have to be rough for lack of detailed information. These results indicate

Table 4: Parameters with dimension for a number of bows and an estimation of a_D.

Ref.	Bow type		$\overline{F}(\lvert\overline{OD}\rvert)$ kgf	$\lvert\overline{OD}\rvert$ cm	$2\overline{m}_b$ kg	$2\overline{L}$ cm	$\overline{\delta}_{bv}$ kgf cm/kg	A_b	a_D
[4]	flatbow	KL	15.5	71.12	0.325	182.9	11000	0.52	0.20
[3]	longbow	KL	46.5	74.6	0.794	187.4	11000	0.52	0.25
[1]	steelbow	KL	17.0	71.12	0.709	168.9	1300	0.52	0.69
[9]	Tartar	PE	46.0	73.66	1.47	188.0	20000	0.59	0.07
[8]	Turkish	TU	69.0	71.12	0.35	114.0	20000	1.1	0.77
[11]	modern	WR	12.6	71.12	0.29	170.3	30000	0.70	0.07

that the short Turkish bow is made of a combination of good materials which are used to the full extent. This explains the good performance of these bows in flight shooting, and not the mechanical performance of these bows; see Table 3. The modern materials are the best, but the calculated value of a_D for the modern bow suggests that they are used only partly. The calculated efficiency of the modern working-recurve bow correlates well with values given in the literature. The value of 72.9% is, however, rather low. The maximum for the efficiency according to Equation (15) from the mass of the string determined by the parameters given in Table 3 is about 90%. Hence, it seems to be possible to improve this quality coefficient with the help of the mathematical model presented here.

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