Archery and Mathematical Modelling [1]

B.W. Kooi

1 Introduction

One way of studying ancient bows is to make replicas and use them for experiments. In the present paper the emphasis is on a different approach, the use of mathematical models. Such models permit theoretical experiments on computers to gain insight into the performance of different types of bow. The use of physical laws and measured quantities, such as the specific mass of materials, in constitutive relations yields mathematical equations. In many cases the complexity of the models obtained will not permit the derivation of the solutions by paper and pencil operations. Computers can then be used to approximate the solution. However, even this procedure will mostly necessitate simplifications. Sometimes essential detailed information is missing. In other situations assumptions need to be made to keep the model manageable. In that case the model has to be validated by the comparison of predicted results with actually measured quantities to justify the assumptions. For that purpose, fortunately, replicas can be employed.

Mathematical models must accommodate all quantities which determine the action of the bow. Such quantities are often called design parameters. Calculations are possible only if all the parameters are known. Descriptions of bows in the literature are often incomplete, so that comprehensive evaluation becomes impossible.

Theoretical experiments with models consists to a large extent of the research on the influence of the design parameters on the performance of the bow. This presupposes definition of good performance which fits the context of interest. Flight shooters are only interested in a large initial velocity. For target archery, on the other hand, the bows has to shoot smoothly and steadily.

In the 1930's bows and arrows became the object of study by scientists and engineers; see Hickman, Klopsteg and Nagler[1,2]. Their work influenced strongly the design and construction of the bow and arrow. Experiments were performed to determine the influence of different parameters. Hickman made a very simple mathematical model for flatbows. Later Schuster[3] and Marlow[4] also developed mathematical models to describe the mechanical action of a bow. Schuster dealt with the ballistic of the modern, so called working-recurve, bow. Schuster's model has the strange feature that bows appear to have 100% efficiency. Marlow introduced an elastic string in the model in order to explain this discrepancy with reality.

The description of our mathematical model is beyond the scope of this paper. The reader is referred to papers[5–7]. The developed model is much more advanced, so that more detailed information is obtained. This gives a better understanding of the action of rather general types of bow. Elsewhere[8,9] we have shown how this model can be adapted for the description of the ballistics of a modern bow. The predicted efficiency is smaller than 100%

[1] *Journal of the Society of Archer-Antiquaries* **34**:21-29 (1991)

because in this model part of the available energy remains in the limbs and string and is not transferred to the arrow.

This model was validated by a comparison of the measured initial velocity of an arrow shot with a modern bow with the predicted value[10]. As part of the Mary Rose project[11] the measured weight of a replica was correlated with the predicted value. In both cases the predictions were sufficiently good.

The aim of the present paper is to use the model for an evaluation of the performance of bows used in the past and in our time. We try to uncover the function of the siyahs or ears of the Asiatic composite bow and to find the reason for the differences in the performance of the longbow and the Turkish bow in flight shooting.

2 Mathematical modelling

In essence the bow proper consists of two elastic limbs, often separated by a rigid middle part, the grip. The bow is braced by fastening a string between both ends of the limbs. After an arrow is set on the string the archer pulls the bow from braced situation into full draw. This completes the static action in which potential energy is stored in the elastic parts of the bow. After aiming, the arrow is loosed or released. The force in the string accelerates the arrow and transfers part of the available energy as kinetic energy to the arrow. Meanwhile the bow is held in its place and the archer feels a recoil force in the bowhand. After the arrow has left the string the bow returns to the braced position because of damping.

As stated before, a complete description of the mathematical model is beyond the scope of this paper. An extensive discussion is presented elsewhere[5–9]. A summary of all important quantities in the model which determine the mechanical action of the bow is listed below:

bow	length of the limbs
	length of the grip
	shape of the unstrung limbs
	shape of cross-section of the limbs at all positions along the limbs
	elastic properties of the materials of the limbs
	specific mass of the materials of the limbs
	shape and mass of the ears, if these are present
	mass of the horns
	fistmele
	draw length
string	mass of the string
	elastic properties of the string
arrow	mass of the arrow

These quantities, the design parameters, determine the weight of the bow. In practice the bowyer tillers the bow such that it has finally the desired weight for a particular draw length. The archer on the other hand sets the fistmele by the adjustment of the length of the string.

For flight shooting the initial velocity of the arrow leaving the string is very important. The larger this velocity the larger the maximum attainable distance. The actual distance depends also on the elevation angle (nearly 45°) and the drag of the arrow in the air. A requirement for target shooting and hunting is that the bow shoots smoothly. It is difficult to translate this feature into mathematics. High efficiency is a good criterion. However, a heavy arrow always yields a high efficiency and, unfortunately so, a small initial velocity and therefore a short distance. Hence, we have a combination of factors. The recoil-force, i.e. the force, the archer feels in the bowhand after the release, also seems to be important. The way this force changes in time can be calculated with the model, but it cannot be summarized by a single number.

The bow should not exaggerate human error. To assess the sensitivity of the bow, its performance is calculated repeatedly with slightly different values for the design parameters. If the performance depends strongly on a design parameter, the archer has to take care that the value of this parameter is as constant as possible. To achieve this archers need skill besides technique.

3 Validation of the mathematical model

Mathematical models may be beautiful by themselves and the way to solve them interesting, but they should mimic the mechanical action of the bow and arrow good enough if they are used in the design of a bow or a sensitivity study.

We checked static action by comparing the measured weight of a replica of longbows found on the recovered Mary Rose with calculated values. The Mary Rose was Henry VIII 's warship which sank in 1545 in The Solent, a mile outside Portsmouth. She was recovered in 1982 with 139 yew longbows. Tests with these bows have demonstrated that while it is possible to string and draw the bows to 30 inch, considerable degradation within the cell structure of the wood has prevented a realistic assessment of the original weight. A replica was made by Roy King, bowyer to the Mary Rose Trust. Prof. P. Pratt, Imperial College of Science & Technology London, measured all parameters which are required to calculate the mechanical performance of a bow. The weight of this replica was also measured. It compared very well with the predicted value calculated with the mathematical model (differences within 1%)[11]. These results imply that if a good estimate of the original modulus and density can be obtained, the original mechanical performance of the longbows can be calculated from the dimensions of these recovered bows.

Data obtained with the test set-up described extensively elsewhere[10], permitted a comparison of predicted and measured arrow velocities. The dynamic action of bows could be checked in this way. We used a modern bow made of maple in the core and glass fibres embedded in strong synthetic resin at both sides of the core. All the essential parameters listed above were measured. We measured the density and elastic modulus of both the wood and the fibreglass and at a number of stations along the limbs the shape of the cross-

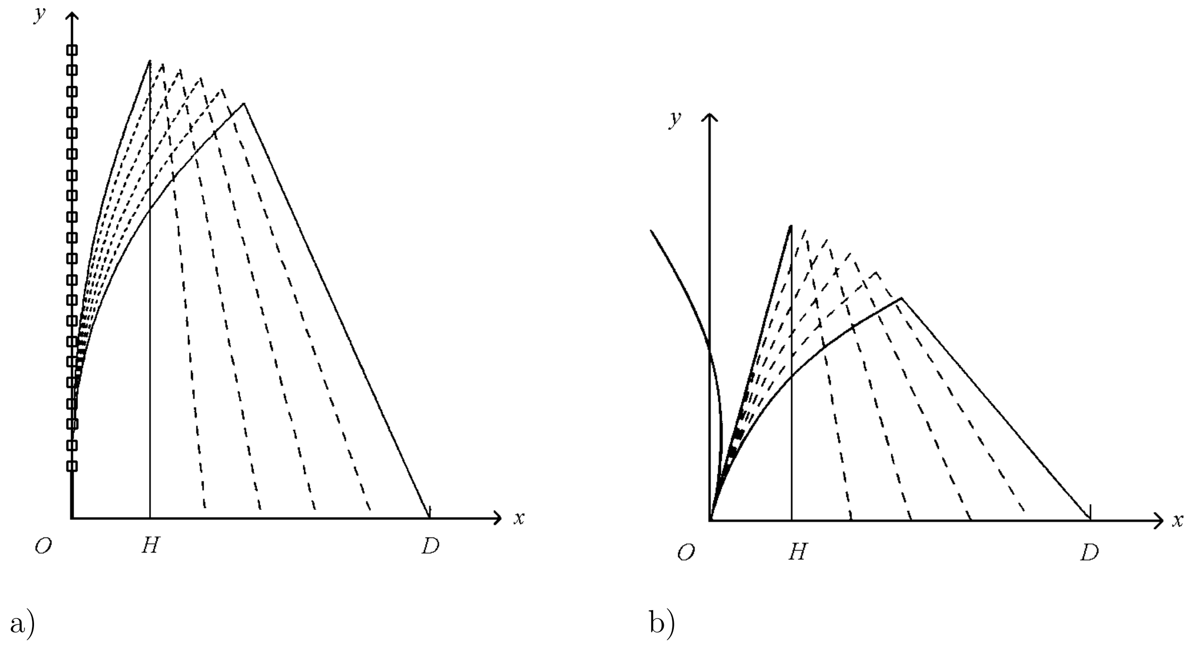

Figure 1: Non-recurve bows: Static deformation shapes (a) of the KL-bow and (b) of the AN-bow.

sections. The results were used to determine the bending properties of the limbs. Finally the elastic modulus and the mass of the string were measured.

The predicted weight was too high and therefore a knockdown factor was used for the bending stiffness of the limbs, so that the calculated weight became equal to the measured value. The predicted amount of energy stored in the bow by drawing it from the braced situation to full draw, differed only slightly from the measured value. The measured efficiency was a few percent below the calculated value. In the model internal and external damping are neglected. This explains part of the discrepancy.

4 Classification of the bow

The classification of the bows we use is based on the geometrical shape and the elastic properties of the limbs. The bows of which the upper half is depicted in Figure 1 are called non-recurve bows. In the model the bow is assumed to be symmetric with respect to the line of aim. So we need to deal with only one half of the bow. These bows have contact with the string only at their tips. In the case of the static-recurve bow, see Figure 2, the outermost parts of the recurved limbs (the ears) are stiff. In the braced situation the string rests on string-bridges. These string-bridges are fitted to prevent the string from slipping past the limbs. When such a bow is drawn, at some moment the string leaves the bridges and has contact with the limbs only at the tips. In a working-recurve bow the limbs are also curved in the 'opposite' direction in the unstrung situation, see Figure 3. The parts of a working-recurve bow near the tips, however, are elastic and bend during the final part of the draw. When one draws such a bow, the length of contact between string and limb decreases gradually until the point where the string leaves the limb coincides with the

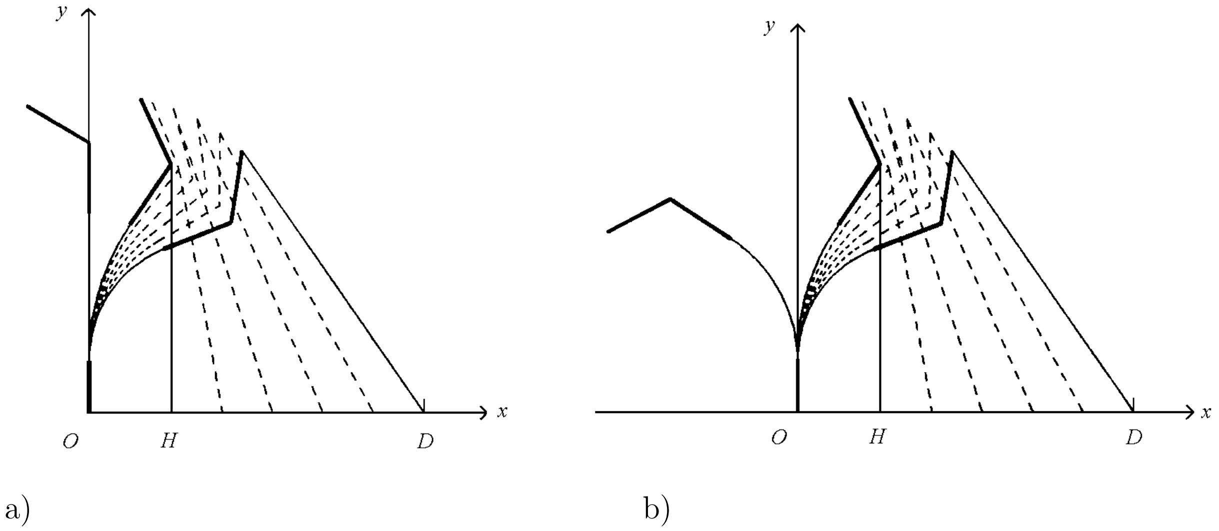

Figure 2: Static-recurve bows: Static deformation shapes (a) of the PE-bow and (b) of the TU-bow.

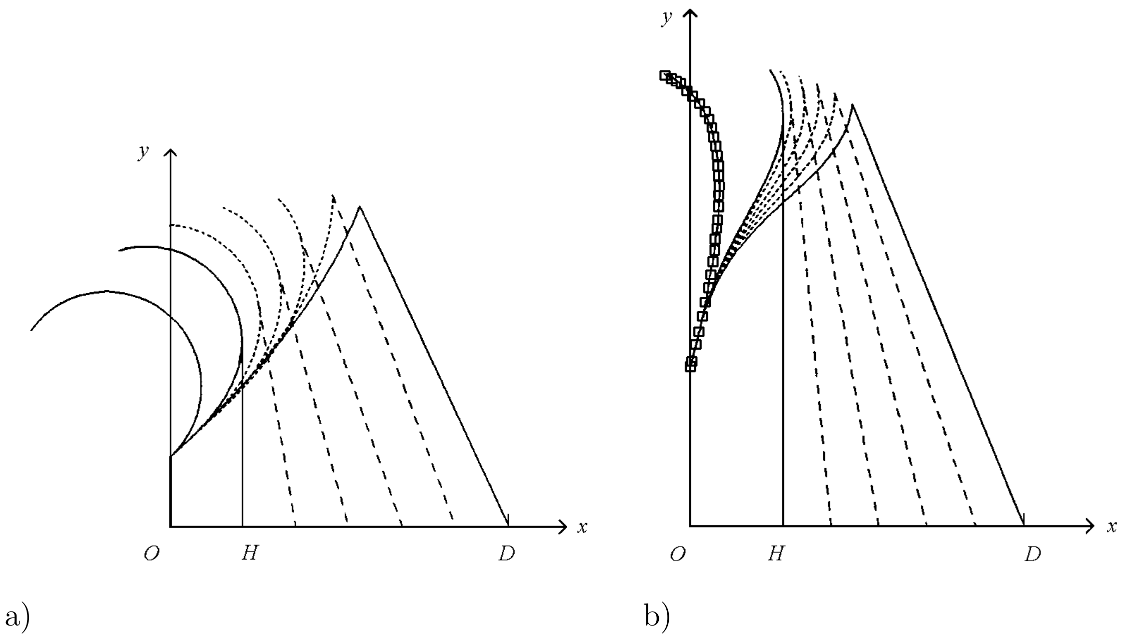

Figure 3: Working-recurve bows: Static deformation shapes (a) of the ER-bow and (b) of the WR-bow.

Table 1: Quality coefficients for various bows. Note that the values for the working recurve WR-bow are adapted for the masses of the arrow and string but the stiffness of its string is about twice those of the other bows.

Bow	q	η	ν
KL-bow	0.407	0.765	2.01
AN-bow	0.395	0.716	1.92
PE-bow	0.432	0.668	1.94
TU-bow	0.491	0.619	1.99
ER-bow	0.810	0.417	2.08
WR-bow	0.434	0.770	2.09

tip. The string remains in that position during the final part of the draw. Elsewhere[5] we dealt with the statics (before arrow release) of these three types of bow. We studied the dynamics (after arrow release) of the non-recurve bow[6], the dynamics of the static-recurve bow[7] and finally that of the working-recurve bow[8].

In the model the action of a bow and arrow combination is fixed by one point in a high dimensional parameter space. Representations of different types of bow used in the past and in our time form clusters in this parameter space. We study the performance of different types of bow and start with a straight-end bow described by Klopsteg[1]. This bow is referred to as the KL-bow. The shape of the KL-bow for various draw lengths is shown in Figure 1(a). The AN-bow represents another non-recurve bow, the Angular bow found in Egypt and Assyria. The shape of the unstrung bow, shown in Figure 1(b), implies that in the braced situation the limbs and the string form the characteristic triangular shape. We consider two static-recurve bows, one from China, India and Persia, to be called the PE-bow, and one which resembles a Turkish flight bow, to be called the TU-bow. The shapes of these bows for various draw-lengths are shown in Figure 2. One of the working-recurve bows, to be called the ER-bow, possesses an excessive recurve. It resembles a bow described and shot by Hickman[1]. The other working-recurve bow is the modern one which was used for the validation of the model[10]. This bow shown in Figure 3(b), is referred to as the WR-bow.

Three quality coefficients for these types of bow are shown in Table 1. These coefficients are defined for equal weight, draw length and mass of the limbs. Moreover the mass of the arrows and strings were the same for all reported bows. This makes an honest comparison possible. Unfortunately the stiffness of the string of the WR-bow is about twice that of the other bows. The static quality coefficient q measures how much recoverable energy is stored in the fully drawn bow. It is defined as the additional deformation energy stored

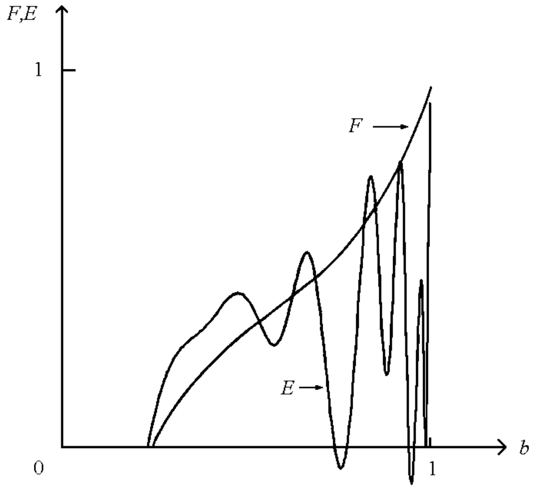

Figure 4: Static (F) and dynamic (E) force-draw curves for the static-recurve PE-bow.

in the elastic limbs and string by drawing the bow from the braced into the fully drawn position divided by the weight timed the draw length. The efficiency η is the kinetic energy transferred to the arrow divided by the just mentioned additional deformation energy. So, it is the part of the available amount of energy which is transferred to the arrow as useful energy. The third quality coefficient ν is proportional to the initial velocity. The constant depends only on the weight, draw length and mass of the limbs.

The static quality coefficient is 1 when the draw-force is uniformly equal to the weight for all draw lengths for a fictitious bow with no fistmele. Just as the efficiency, this coefficient gives the actual value relative to a basic, characteristic value. The results show that in practice q is slightly smaller that 0.5 except for the ER-bow with the extreme recurve.

In what follows we will comment on a statement made by Hamilton[12]:

> "The function of the "ears", or siyahs, is well known today and no one can question the superiority of the type of bow which still holds the world record of shooting an arrow 972 yards (Klopsteg[2])".

Hamilton continues:

> "The siyah contributes in three ways to improve cast in the arrow".
>
> "(1) It provides leverage for the bowstring so the bow can be designed to approach maximum weight earlier in the draw allowing more energy to be stored for the cast".

This statement is in agreement with our results. The static quality coefficient of the PE-bow is larger then of the straight-end KL-bow.

In Figure 4 the static and dynamic force draw curve are shown for the PE-bow. The line indicated with F shows a bend at the place where the string leaves the bridges. The TU-bow stores even more energy in the fully drawn position, obviously because of the recurve of the working part of the limbs. So the good static performance of flight bows may result only partly from the use of the stiff ears.

> "(2) Upon release, the bowstring imparts its energy to the arrow more uniformly and at a higher and more sustained rate of thrust than in a ordinary bow without siyahs."

This statement is not supported by the results obtained with the model. Because of the relatively heavy ears, there is a sudden decrease in the force in the string and, by implication, in the acceleration force upon the arrow. The result of this is oscillatory behaviour as shown in Figure 4. Consequentially the efficiency of static-recurve bows a rather low. The amplitude of the oscillations depends largely on the modulus of elasticity of the string and the mass of the arrow relative to the mass of the ears.

> "(3) When the bow string reaches the bridges it is in effect shortened, increasing the tension again on the bowstring and giving the arrow a final snap as it leaves the bow."

The dynamic force draw curve (E in Figure 4) shows that the acceleration of the arrow is rather large when the string has contact with the bridges.

Notwithstanding this, the efficiency η of the PE-bow and certainly that of the TU-bow, is rather low. This implies that the initial velocity ν is not as large as one would expect on the basis of the static performance. This is caused by the relative heavy ears. These considerations demonstrate why these bows can, inherently, not be better than long straight-end bows. A large part of the available energy remains in the vibrating limbs and string after arrow exit.

This holds even to a larger extent for the ER-bow. The string cannot slow down the now light ends of the limbs during the final part of the acceleration of the arrow when the bow is close to its braced situation again.

The modern WR-bow seems to be a good compromise between the non-recurve bow and the static-recurve bow. The recurve yields a good static quality coefficient and the light tips of the limbs give a reasonable efficiency.

5 Construction of the bow

But what made the Turkish flight bow a superb type of bow for flight shooting? Until now we dealt with the mechanics of the bow but not with its construction. The efficiency is greatly affected by the relative mass of the arrow relative to that of the limbs. For a fixed mass of the arrow, the lighter the limbs the better the efficiency. This is the item where technology becomes important. The minimum mass of the limbs for a fixed weight and draw is determined largely by the appropriateness of the material to store energy.

In the past man used bows which differ not only in shape but also in the materials applied. Simple bows made out of one piece of wood, straight and tapering towards the ends have been used by primitives in Africa, South America and Melanesia. In the famous English longbow the different properties of sapwood and heartwood were deliberately put to use. Eskimoes used wood together with cords plaited of animal sinews and lashed to the wooden core at various points. The Angular bow found in Egypt and Assyria are examples

Table 2: Mechanical properties and the energy per unit of mass referred to as $\overline{\delta}_{bv}$ for some materials used in making bows[13].

material	working stress $kgf/cm^2 \times 10^2$	elastic modulus $kgf/cm^2 \times 10^5$	specific mass $kg/cm^3 \times 10^{-6}$	$\overline{\delta}_{bv}$ kgf cm/kg
steel	70.0	21.0	7800	1300
sinew	7.0	0.09	1100	25000
horn	9.0	0.22	1200	15000
yew	12.0	1.0	600	11000
maple	10.8	1.2	700	7000
glassfibre	78.5	3.9	1830	43000

of composite bows. In these bows more than one material was used. In Asia the bow consisted of wood, sinew and horn; "Just as man is made of four component parts (bone, flesh, arteries and blood) so is the bow made of four component parts. The wood in the bow corresponds to the skeleton in man, the horn to the flesh, the sinew to the arteries, and the glue to the blood." These bows were used by the Mongolian races of Eastern Asia. They reached their highest development in India, in Persia and in Turkey. In modern bows maple and glass or carbon fibres, embedded in strong synthetic resin are used.

In Table 2 indications of the mechanical properties are given for some materials used in making bows. From this table we conclude that it is possible to store much more energy per unit of mass in the materials of the composite bow, sinew and horn, than in wood, the material of the old simple bow. Moreover, in the composite bows not only better materials were used, but they were also used in a more profitable manner. Sinew is very strong in tension. It is therefore used on the back side. Horn withstands compression very good. It is applied on the belly side of the limbs. Hence, a composite bow with the same mass as a simple wooden bow can have a much larger weight. This explains the good performance of the composite flight bow in flight shooting.

In Table 3 we give values for the weight, draw, mass and length, for a number of bows described in the literature. The longbow is the replica of the Mary Rose bow. The calculated weight of this bow called MRA1 was 102.4 lbs. If the same values are used for the material properties of the Mary Rose bow called A812, the estimated weight becomes 108 lbs. W.F. Paterson[14,15] also investigated this bow. Dr. Clarke calculated that the draw weight would be about 76 1/2 lb, depending on the modulus of elasticity of the yew[15]. The late Paterson in a letter to the author informs that Dr. Clarke: "admits an error by the factor of two in his calculations. His estimate should now read 153 lb". Unfortunately no value for the elastic modulus is mentioned[14,15], but the final discrepancy is probably caused

Table 3: Parameters for a number of bows and an estimation of $\overline{d}_{bv}$ the weight times draw length divided by the mass of one limb. For comparison also the estimated values for $\overline{\delta}_{bv}$ are given. When materials are used to their full extend, $\overline{d}_{bv}$ divided by about 4 equals $\overline{\delta}_{bv}$.

Ref.	Type	weight kgf	draw cm	mass kg	length cm	$\overline{\delta}_{bv}$ kgf cm/kg	$\overline{d}_{bv}$ kgf cm/kg
[1]	flatbow	15.5	71.12	0.325	182.9	9000	6800
[11]	longbow	46.5	74.6	0.794	187.4	9000	8700
[16]	steelbow	17.2	71.12	0.709	168.9	1300	3500
[17]	Tartar	46.0	73.66	1.47	188.0	20000	4600
[18]	Turkish	69.0	71.12	0.35	114.0	20000	28000
[10]	modern	12.6	71.12	0.29	170.3	30000	6200

by a difference in this mechanical property. We decided to use a value of 0.75 10^5 kgf/cm^2. Parenthetically, the spread in the modulus of elasticity of yew yields makes the predictions of the weight (almost proportional to the modulus of elasticity of yew) of the Mary Rose bows uncertain.

The quantity denoted by $\overline{d}_{bv}$ is proportional to the amount of energy stored in the bow per unit of mass. It equals the weight times draw length divided by the mass of one limb. When materials are used to their full extend $\overline{d}_{bv}$ divided by about 4 should equal $\overline{\delta}_{bv}$.

We saw that because of the stiff ears or a recurve of the working parts of the limbs, much energy is stored in the static-recurve bow. In a recurved bow the amount of energy in the braced position is already large. This implies that the limbs must be relative heavy in order to store this extra and not usable energy, in addition to the recoverable energy. This is the price paid for a larger static quality coefficient. On the other hand, sinew and horn are relative tough and flexible materials, see Table 2. This explains why the use of these materials fits well with the recurved shape of the unstrung bow. The values in Table 3 show that the Turkish bow is very strong but also light. This indicates why it permits one to shoot a light arrow a long distance. A short bow is moreover easier in operation and is suited for the use on horse back. In a letter[19] to the author E. McEwen informs that: "Pope did not properly test his larger 'Tartar' (actually Manchu-Chinese) bow. He only drew it 36 inches and bows of this type and size are made to draw as much as 40 inches". Pope only mentions the weight for a draw length of 29 inches. If the weight of this bow with a draw length of 101.6 cm is 70 kgf, we have $\overline{d}_{bv} = 9700$ kgf cm/kg. This value is still rather low and this means that the materials of this bow are used only partly. This supports McEwens[19] view that: "this bow was probably a 'test' bow used for exercise and for the military examinations and not meant for actual shooting". The values obtained for the straight-end bows look very realistic. In the modern bow there is a surplus of material near the riser section. This affects the efficiency only slightly. For, this part of the limb moves hardly and therefore the involved kinetic energy is small. In this bow there is also a rather large amount of unrecoverable energy in the braced position. This puts a constraint on the amount of recurve. With respect to this, it is perhaps more important that the efficiency of working-recurve bows decreases with increasing recurve.The mechanical properties of the materials of these bows, however, are much better than those of the ancient composite bows. Indeed, the modern bow holds now the longest flight shooting record.

Additional features were added to improve the performance especially for target shooting; relative immunity of the mechanical properties to temperature and humidity variations, no tendency to follow the string, use of stabilizers, sculptured long centre-shot riser section, bow sights and last but not least stronger materials for bowstrings. Finally an improved arrow design adds to the steadiness of the equipment.

6 Conclusions

We conclude that these results indicate that the initial velocity is about the same for all types of bow under similar conditions. So, within certain limits, the design parameters which determine the mechanical action of a bow arrow combination appear to be less important than is often claimed. We would endorse a view one could call holistic. It is not always possible to isolate a single feature and state that it solely accounts for a good or bad performance of the whole bow, as Hamilton[12] did. Rausing[20] studies the development of the composite bow. According to him, the fact the static quality coefficient of the short static-recurve bow to be larger than that of the short straight bow, disposes of the statement of Pitt-Rivers, Balfour and Clark: "the composite bow has no inherent superiority over the wooden self-bow, so long as the latter was made from the most favourable kinds of timber and expertly used". The results obtained with the mathematical model suggest that, if the word inherent has the meaning within the context we used it in Section 4, their statement is true. A combination of many technical factors made the composite flight bow better for flight shooting.

The quality coefficients of the modern bow are only slightly better than those of the other types of bow. Materials used in modern working-recurve bows can store more deformation energy per unit of mass than the materials used in the past. Moreover the mechanical properties of these materials are more durable and much less sensitive to changing weather conditions. This contributes most to the improvement of the modern bow.

We hope that we have shown that mathematical modelling can be a helpful tool in the research on archery, not only for the design of new bow equipment but also for understanding the development of the bow in the past.

References

[1] C. N. Hickman, F. Nagler, and P. E. Klopsteg. *Archery: the technical side.* National Field Archery Association, Redlands (Ca), 1947.

[2] P.E. Klopsteg. *Turkish archery and the composite bow.* Simon Archery Foundation, Manchester, 1987.

[3] B. G. Schuster. Ballistics of the modern-working recurve bow and arrow. *Am. J. Phys.*, 37:364–373, 1969.

[4] W. C. Marlow. Bow and arrow dynamics. *Am. J. Phys.*, 49:320–333, 1981.

[5] B. W. Kooi. *On the Mechanics of the Bow and Arrow.* PhD thesis, Rijksuniversiteit Groningen, 1983.

[6] B. W. Kooi and J. A. Sparenberg. On the static deformation of a bow. *Journal of Engineering Mathematics*, 14:27–45, 1980.

[7] B. W. Kooi. On the mechanics of the bow and arrow. *Journal of Engineering Mathematics*, 15:119–145, 1981.

[8] B. W. Kooi. The 'cut and try' method in the design of the bow. In H. A. Eschenauer, C. Mattheck, and N. Olhoff, editors, *Engineering Optimization in Design Processes*, volume 63 of *Lecture Notes in Engineering*, pages 283–292, Berlin, 1991. Springer-Verlag.

[9] B. W. Kooi. On the mechanics of the modern working-recurve bow. *Computational Mechanics*, 8:291–304, 1991.

[10] C. Tuijn and B. W. Kooi. The measurement of arrow velocities in the students' laboratory. *European Journal of Physics*, 13:127–134, 1992.

[11] R. Hardy. *Longbow.* Mary Rose Trust, Portsmouth, 1986. pp. 4-6.

[12] T. M. Hamilton. *Native American Bows.* Missouri Archaeological Society, Columbia (Ms), 1982.

[13] J. E. Gordon. *Structures or Why Things Don't Fall Down.* Pelican, Harmondsworth, 1978.

[14] W. F. Paterson. "Mary Rose" - a preliminary report. *Journal of the Society of Archer-Antiquaries*, 23:29–34, 1980.

[15] W. F. Paterson. "Mary Rose" - a second report. *Journal of the Society of Archer-Antiquaries*, 24:4–6, 1981.

[16] R. P. Elmer. *Target Archery.* Hutchinson's Library of sports and pastimes, London, 1952.

[17] S. T. Pope. *Bows and Arrows.* University of California Press, Berkeley (CA), 1974.

[18] Sir R. Payne-Gallwey. *The Crossbow.* Holland Press, London, 1976.

[19] E. McEwen. Private communications, 1991.

[20] G. Rausing. *The Bow.* Acta Archaeologica Lundensia. CWK Gleerups Förlag, Lund, Sweden, 1967.