THE ANALYSIS ON "S-CURVE MOTION" OF ARROW IN ARCHERY SPORTS

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ABSTRACT

In this paper, the multirigid body spring hinge system model of an arrow arm is set up, based on which its dynamical buckling characteristic equation and post-buckling equilibrium equation, when it is sent out, are deduced. Also, its nonlinear dynamic responses in motion is analyzed.

NOMENCLATURE

L, H, J_{fx}, J_{fy}	relation matrix
X, Y	coordinate column
K_i	stiffness coefficient
l_i	length of i^{th} element
φ_i	angle between the element axis and the positive direction of X axis
θ_i	the variation of φ_i
D_*	constraints matrix
ϕ_*, F^*, T^*, T^*	generalized force matrix
K^*	generalized stiffness matrix
I	the matrix for mass moment of inertia
μ, μ_*^*	generalized mass matrix
ξ, ξ^*	generalized damping force matrix
$\lceil \ \rfloor$	diagonal matrix

1. INTRODUCTION

In this paper, the "S-curve Motion" phenomenon of arrow arm, that is the "Snake Motion", is analyzed, based on the multirigid-body discrete method. At present, archery is a popular sport. Usually, an experienced sportsman can find the phenomenon of "knocked arm". The pictures from high speed photograph showed that the arrow arm displays bending deformation when it is shooting, moreover it assumes a "S-curve Motion" in its flight, to which, up to now, there is no reasonable explanation. Based on the arrow arm's multirigid-body discrete model, it is shown that arrow arm is subjected to dynamical instability in motion due to its great accelerator.

Furthermore, the large deformation curve is regarded as the vibration initial condition departing from bow string, and its dynamical equation is used to calculate the large deformation nonlinear vibration components. The equation is a two ordered nonlinear differential one for the system with multi-degrees of freedom. It seems that the "S- curve motion" of arrow comes from the combination of the rigid motion components and nonlinear vibration components with respect to motion coordinate.

2. BUCKLING AND POST-BUCKLING ANALYSIS OF BAR

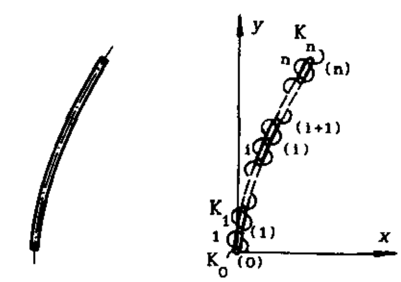
The bar that has plane curve is cut into N straight segments with equivalent section area, which is shown in Fig(1). And every segment is regarded as a element rigid. The relation between two element rigid assumes joint or bending spring, the equivalent intensive rigidity is provided in[3].

Suppose the corner between the axis for the i^{th} element rigid and the positive direction of the X-axis to be ϕ_i. X_i and Y_i is the coordinate of the node i, then

$$\{X\}=[H]\{\cos\phi\} \tag{1}$$
$$\{Y\}=[H]\{\sin\phi\} \tag{2}$$

Where

$$[H]=\begin{bmatrix} l_1 & 0 & \cdots\cdots & 0 \\ l_1 & l_2 & \cdots\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ l_1 & l_2 & \cdots\cdots & l_n \end{bmatrix}$$



(a) (b)

Fig.1.The Multi Rigid Body Discrete Model For Bar

Differienize Eq(1) and (2), we have

$$\{\delta X\}=-[H]\lceil \sin\phi \rfloor\{\delta\phi\} \quad (3)$$
$$\{\delta Y\}=[H]\lceil \cos\phi \rfloor\{\delta\phi\} \quad (4)$$

The total potential energy of the system Π is the algebraic sum of its strain energy U and the net potential energy V,

$$\Pi=U+V \quad (5)$$

The strain energy of the system shown in Fig. 1 is as follows:

$$U=(1/2)\{\theta\}^T[K]\{\theta\} \quad (6)$$

The net potential energy of the system is:

$$V=-\{X\}^T\{F_x\}-\{Y\}^T\{F_y\}-\{\theta\}^T\{M\}$$
$$+\{X_0\}^T\{F_x\}+\{Y_0\}^T\{F_y\} \quad (7)$$

where the system is shown in Fig.1., and $\{\theta\}$ is the variation of $\{\phi\}$. The detailed deduction of [K],U,V refers to [3].

According to energy criterion of the system's equilibrium stability with finite degrees of freedom, we have

$$\{\delta\ \theta\}^T([k]\{\theta\}+\lceil \sin\phi \rfloor[H]^T\{F_x\}$$
$$-[\cos\phi][H]^T\{F_y\}-\{M\})=0 \quad (8)$$

Usually, as a structure has various bearing, $\{\theta\}$ is not independent generalized coordinate, among which constraints exist as $f_i(\theta)=0(i=1,\ldots,m)$.

For this reason, the constraint matrix [D] is turned to remove the effect of the constraints. The determination of [D] refers to [3].

Take $\{q\}=[\theta_1\ \ \theta_2\ldots\ldots\ \ \theta_{n-m}]^T$ as the independent generalized coordinates, then the variational relation between {q} and $\{\theta\}$ is as follows:

$$\{\delta\theta\}=[D]\{\delta q\} \quad (9)$$

thereupon, the Eq.(8) is changed into Eq(10):

$$[D]^T([k]\{\theta\}+\lceil\sin\phi\rfloor[H]^T\{F_x\}$$
$$-\lceil\cos\phi\rfloor[H]^T\{F_y\}-\{M\})=\{0\} \quad (10)$$

The deformation of bar situated in a small domain when it is buckling, so

$$[D(\phi_0)]=[D_0],\quad \{\theta\}=[D_0]\{q\} \quad (11)$$

For the straight bar, when the initial angle the bar's axis inclining to the positive direction of X axis is equal to zero, we have,

$$\lceil\sin\phi\rfloor=\lceil\theta\rfloor \quad \lceil\cos\phi\rfloor=[E] \quad (12)$$

clearly

$$[D_0]^T([k]\{\theta\}+\lceil\theta\rfloor[H]^T\{F_x\}-[H]^T\{F_y\}-\{M\})=\{0\} \quad (13)$$

When the straight bar is situated in the critical state, $\{\theta\}=0$, we have

$$([D_0]^T[k][D_0]+[D_0]^T[T_x][D_0])\{q\}=\{0\} \quad (14)$$

Where

$$\lceil\theta\rfloor[H]^T\{F_x\}=[T_x]\{\theta\}$$
$$[T_x]=\sum_{i=1}^{n}[J_i]F_{xi}$$
$$[J_i]=\mathrm{diag}[l_1\ l_2\ \ldots\ldots\ l_i\ 0\ldots\ldots0]_{n\times n}$$

The eq(14) is the buckling characteristic equation of straight bar.

When the staff is situated in post-buckling, its deformation isn't minute, the a small deformation assumption isn't still suitable, hence

$$\{\delta\ \theta\}^T([k]\{\theta\}+[T_x]\{\sin\theta\}-[T_y]\{\cos\theta\}-\{M\})=0 \quad (15)$$

Assume that, the staff has S bearing in X direction and R bearing in Y direction, then

$$[J_{fx}]\{\cos\theta\}=\{C_x\} \quad (16)$$
$$[J_{fy}]\{\sin\theta\}=\{C_y\} \quad (17)$$

Where

$$[J_{fx}]=\begin{bmatrix} l_1 & l_2 & \cdots & l_{11} & 0\ldots0 & 0 & \cdots & \cdots & \cdots 0 \\ l_1 & l_2 & \cdots & \cdots & \cdots & l_{12} & 0\ldots0 & 0 & \ldots 0 \\ \cdots & & & & & & & & \cdots \\ l_1 & l_2 & \cdots & \cdots & \cdots & \cdots & l_{1s} & 0 & \ldots 0 \end{bmatrix}_{s\times n}$$

$[J_{fx}]$ and $[J_{fy}]$ has the same form.

Hence, the constraint matrix [D] of straight bar can be got from Eq.(16) and (17),

$$[K^*]\{\theta\}+[T_x^*]\{\sin\theta\}+[T_y^*]\{\cos\theta\}=\{F^*\} \quad (18)$$

where

$$[K^*]=\begin{bmatrix}[D]^T[K]\\ \\ [0]\end{bmatrix}_{n\times n} \qquad [T_x^*]=\begin{bmatrix}[D]^T[T_x]\\ [0]\\ [J_{fx}]\end{bmatrix}_{n\times n}$$

$$[T_y^*]=\begin{bmatrix}[D]^T[T_y]\\ [J_{fx}]\\ [0]\end{bmatrix}_{n\times n} \qquad \{F^*\}=\begin{bmatrix}[D]^T\{M\}\\ \{C_x\}\\ \{C_y\}\end{bmatrix}_{n\times 1}$$

The Eq(18) is the post-buckling equilibrium equation, it is a high ordered nonlinear ones with multi degrees of freedom with respect to θ.

3. THE DYNAMICAL BUCKLING AND POST-BUCKLING ANALYSIS OF ARROW ARM WHEN IT IS BEING SHOOT

At present, most arrow that is used in archery sport is"Easton Shaft" which is made in the USA, it is shown in Fig. 2a). Take the "1816 Easton Shaft" as an example, the solution on arrow arm's dynamical buckling and post-buckling is explained when it is shooting. The relating parameters on geometry and physics are as follows:

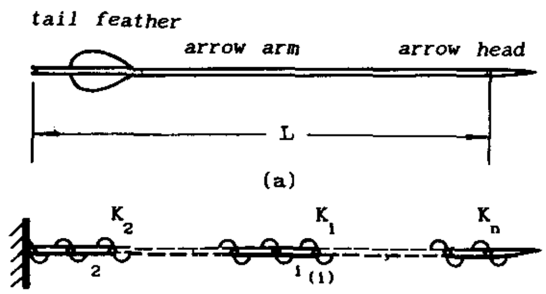


Fig.2. Arrow and Its Multirigid-body Discrete Model

outer diameter of arrow arm	D=18/64 inch
thickness of arrow pipe	t=0.016 inch
mass of arrow head	m_0=4.0g
mass of arrow arm	m_1=20.5g
Young's module of elasticity	E=7.0x10^{10}N/m^2
length of arrow arm	L=700mm

Shooting is a progress in which the arrow is pushed forward and moved accelerately. In the process, the arrow is in a dynamical equilibrium

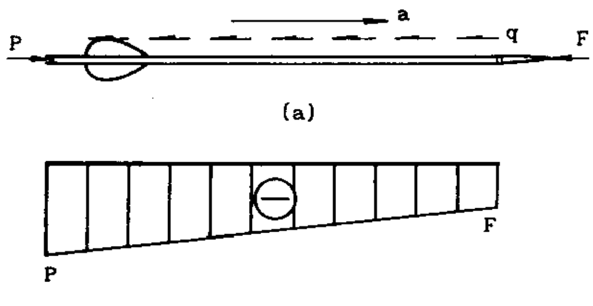


Fig. 3. Arrow's Froce Diagram and Arrow Arm's Axial Diagram

state subjected to the inertia force and elastic restoring force. It is assumed that the arrow makes uniform acceleration in the progress, then the arrow's force diagram and its arm's axial force diagram are shown in Fig.3.

Fig.2b is the arrow arm's multirigid- body discrete model, in which the arrow head is looked as free, and its end is looked as hinge. Its critical acceleration is counted based on the model,

$a_{cr}=928.109m/s^2$

The arrow's shooting time and leaving speed is measured by making use of the advise of measuring speed. For example, the shooting time of "1816 Easton Shaft" is 15ms, and its leaving string speed is 50m/s, based on which, its acceleration is

$a=3333.3333m/s^2$

Apparently, the arrow's mobile acceleration is larger than its critical one, and the arrow arm has lost mobile stability. For this reason, the arrow arm before it departs from bow string is bent.

It can be found from the analysis on arrow arm's post-buckling that when it is in motion arrow arm's deformation curve is shown in Fig.6a, and the function of deformation curve is shown as follow

$$y=-113.2984x^8+331.5849x^7-399.9697x^6+256.0199x^5$$
$$-92.86018x^4+19.87191x^3-2.271958x^2+0.3247857x$$
$$-0.002332804 \qquad (0\le x\le 0.7)$$

4. THE ANALYSIS OF ARROW'S MOTION

Because the arrow has lost its mobile stability before it departs from bow string, and it has great deformation, the great deformation curve can be regarded as its vibration initial condition, as a result, the nonlinear vibration with respect to motion coordinate is generated.

The arrow's geometric nonlinear vibration is subjected to the action of damped force, so its multirigid-body discrete model on vibration can be shown in Fig.4, where every element rigid is equal in length $L_e = l/n$. Denote c_i as its damping coefficient which is dependent on relative speed among elements. Here assume c_i as 0.01, take x_{oi}, y_{oi} as the mass centre coordinate of the i^{th} element-rigid, clearly

$$\{X_c\}=l_e[L]\{\cos\phi\} \qquad (19)$$
$$\{Y_c\}=l_e[L]\{\sin\phi\} \qquad (20)$$

where

$$[L]=\begin{bmatrix} 1/2 & 0 & \dots & 0 \\ 1 & 1/2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1/2 \end{bmatrix}_{n\times n}$$

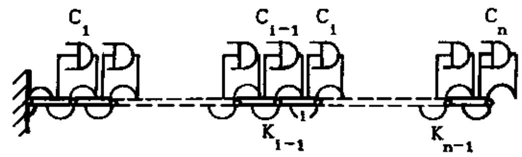


Fig. 4. Arrow's Multirigid-Body Discrete Model

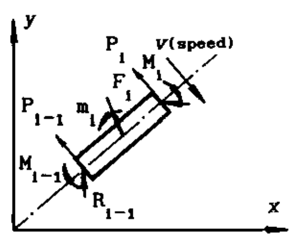


Fig. 5. The i^{th} Element-rigid's Froce Diagram

P_i is the damping force at node i, without considering the air's damping force, the i^{th} element-rigid's dynamical equation is

$$m_c\ddot{X}_{ci}=F_{ix}+R_{ix}+P_{ix}+R_{(i-1)x}+P_{(i-1)x}$$
$$m_c\ddot{Y}_{ci}=F_{iy}+R_{iy}+P_{iy}+R_{(i-1)y}+P_{(i-1)y}$$
$$I_{cei}\ddot{\theta}_i=M_i-M_{i-1}-m_i+M_{pi}+M_{Ri}$$

Here, M_{pi} and M_{Ri} are the main moment of damping force and reaction force with respect to the mass centre, respectively. According to the virtual work principle and by considering the condition on ideal constraints, we have

$$\{\delta X_c\}^T(\lceil M \rfloor\{\ddot{X}_c\}+[C]\{\dot{X}_c\}+\{F_x\})$$
$$+\{\delta Y_c\}^T(\lceil M \rfloor\{\ddot{Y}_c\}+[C]\{\dot{Y}_c\}+\{F_y\})$$
$$+\{\delta\phi\}^T(\lceil I \rfloor\{\ddot{\phi}\}+[K]\{\phi\}+\{M\})=\{0\} \qquad (21)$$

where

$$\{\ddot{X}_c\}=-l_e[L]\lceil \cos\phi \rfloor\lceil \phi \rfloor\{\phi\}-l_e[L]\lceil \sin\phi \rfloor\{\phi\}$$
$$\{\ddot{Y}_c\}=-l_e[L]\lceil \sin\phi \rfloor\lceil \phi \rfloor\{\phi\}-l_e[L]\lceil \cos\phi \rfloor\{\phi\}$$
$$\{\dot{X}_c\}=-l_e[L]\lceil \sin\phi \rfloor\{\phi\}$$
$$\{\dot{Y}_c\}=-l_e[L]\lceil \cos\phi \rfloor\{\phi\}$$

$$[C]=\begin{bmatrix} c_0+c_1 & -c_1 & \cdots\cdots & \cdots\cdots & \cdots & \cdots 0 \\ -c_1 & c_1+c_2 & -c_2 & \cdots\cdots & \cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & -c_{n-1} \\ 0 & \cdots & \cdots & \cdots & -c_{n-1} & c_{n-1}+c_n \end{bmatrix}_{s\times n}$$

Substitute the two ordered derivation of constraint condition to time into the Eq. (21), then, the dynamical equation on arrow arm vibration is

$$[\mu^*]\{\ddot{\phi}\}+[\xi^*]\{\dot{\phi}\}+[k^*]\{\phi\}=\{\Phi^*\} \qquad (22)$$

where

$$[\mu^*]=\begin{bmatrix} [D]^T[\mu] \\ [J_f] \end{bmatrix}_{n\times n} \qquad [\xi^*]=\begin{bmatrix} [D]^T[\xi] \\ [\dot{J}_f] \end{bmatrix}_{n\times n}$$

$$[K^*]=\begin{bmatrix} [D]^T[K] \\ [0] \end{bmatrix}_{n\times n} \qquad \{\Phi^*\}=\begin{bmatrix} [D]^T\{\Phi\} \\ \{0\} \end{bmatrix}_{n\times 1}$$

and

$$[\mu]=l_e^2\lceil \sin\phi \rfloor[L]^T\lceil M \rfloor[L]\lceil \cos\phi \rfloor + l_e^2\lceil \cos\phi \rfloor[L]^T\lceil M \rfloor[L]\lceil \cos\phi \rfloor + \lceil I \rfloor$$

$$[\xi]=(l_e^2\lceil \sin\phi \rfloor[L]^T\lceil M \rfloor[L]\lceil \cos\phi \rfloor + l_e^2\lceil \cos\phi \rfloor[L]^T\lceil M \rfloor[L]\lceil \sin\phi \rfloor)\lceil \dot{\phi} \rfloor + l_e^2\lceil \sin\phi \rfloor[L]^T\lceil C \rfloor[L]\lceil \sin\phi \rfloor + l_e^2\lceil \cos\phi \rfloor[L]^T\lceil C \rfloor[L]\lceil \cos\phi \rfloor + l_e\lceil \cos\phi \rfloor[L]^T\lceil M \rfloor[L]\lceil \cos\phi \rfloor$$

$$\{\Phi\}=l_e\lceil \cos\phi \rfloor[L]^T\{F_y\} - l_e\lceil \sin\phi \rfloor[L]^T\{F_x\}-\{M\}$$

The arrow arm's initial deformation $\{\phi_0\}$ is given by the post-buckling equilibrium equation with $\{\Phi\}=\{0\}$. To the two ordered nonlinear differential equations of multi degrees of freedom system, the segmented linearization approach is used to get the numerical solution which shows the shape of arrow arm at every section, and it is shown in Fig. 6.

Where T is its total flying time,and the unit length of vertical coordinate is three times of that of horizontal's in Fig.6a, the unit length of vertical coordinate is 20 times of that of horizontal's in Fig.6b, c, d. The unit length of vertical coordinate is sixty times of that of horizontal's in Fig.6e.

Obviously, the "S-curve Motion" of arrow is composed of rigid motion components and nonlinear vibration components with respect to motion coordinate.

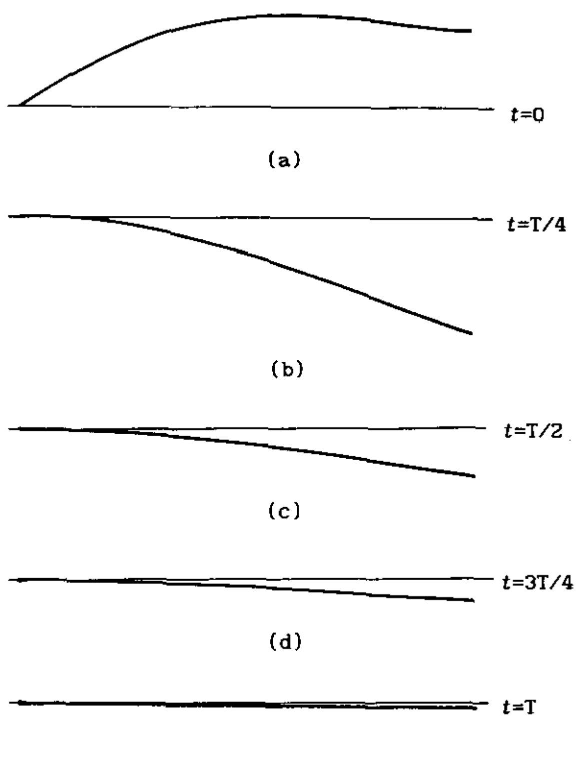


Fig. 6. Arrow's Defromation Curve at t

5. CONCLUSION

The analysis, for the arrow arm's dynamical buckling and post-buckling and its "S-curve Motion", has shown that, the multirigid-body discrete method has high calculation precision, low computation time and simple implementation.

6. REFERENCES

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